

Classes Of Analytic Functions Associated with the
 $E_{(p,q)}^{\lambda,\delta}$ Operator and Their Applications to Third-Order Fuzzy
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**Classes of Analytic Functions Associated with
the $E_{p,q}^{\lambda,\delta}$ Operator and Their Applications to Third-
Order Fuzzy Differential Subordination**

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Abstract

This paper introduces a novel operator $E_{p,q}^{\lambda,\delta}$ defined by means of (p, q) –calculus and investigates its applications in the study of analytic functions. Two new subclasses analytic functions associated with this operator are defined and their coefficient estimates are obtained. The main part of this study is devoted to establishing several results concerning third-order fuzz differential subordination for the defined operator.

Keywords: Analytic functions, (p, q) –calculus, third-order Fuzzy differential subordination, Coefficient Estimates, Starlikeness, Convexity.

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فئات الدوال التحليلية المرتبطة بالمؤثر $E_{p,q}^{\lambda,\delta}$ وتطبيقاتها على التبعية التفاضلية الضبابية من الرتبة الثالثة

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الملخص

تقدم هذه الورقة مؤثرًا جديدًا يُرمز له بالرمز $E_{p,q}^{\lambda,\delta}$ ، مُعرَّفًا باستخدام (p, q) —حساب التفاضل والتكامل ودراسة تطبيقاته على الدوال التحليلية. تم تعريف فئتين فرعيتين جديدتين الدوال التحليلية من المرتبطة بهذا المؤثر وتُستخرج تقديرات لمعاملاتها. وقد خصَّص الجزء الرئيسي من هذه الدراسة لإثبات عدة نتائج تتعلق بالتبعية التفاضلية الضبابية من الرتبة الثالثة لهذا المؤثر المُعرَّف.

الكلمات المفتاحية: الدوال التحليلية، (p, q) —حساب التفاضل والتكامل، التبعية التفاضل الضبابي من الرتبة الثالثة، تقديرات المعاملات، النجمية، التحذب.

1. Introduction

The geometric theory of analytic functions has been one of the most important areas in Complex Analysis since the beginning of the 20th century. The early work focused on univalent functions, coefficient problems and geometric properties such as starlikeness and convexity. A major development happened when Miller and Mocanu introduced the theory of differential subordination [6]. This theory gave a unified method to study many subclasses of analytic functions. Later, researchers extended this idea to second order and third order differential subordinations to obtain sharper and more general results. The third order case is very important because it involves the third derivative and gives better control ([3],[7],[13]). Zadeh introduced the concept of fuzzy sets [15]. The main idea was to allow partial membership instead of classical yes or no. When this idea was applied to Complex Analysis, it led to fuzzy differential

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subordination. The fuzzy approach is more flexible because it measures the degree of subordination. Many authors used this concept to study different classes of analytic functions [9]-[11]. At the same time, the field of q -calculus and (p, q) -calculus has grown rapidly. Jackson was one of the first to work on q -derivatives and q -integrals. Recently, Srivastava introduced (p, q) -calculus as a two-parameter generalization of q -calculus. This calculus reduces to q -calculus when $p = 1$ and to classical calculus when $p = 1$ and $q \rightarrow 1$. Many researchers have used q - and (p, q) -calculus to define new operators in Geometric Function Theory because it gives more freedom to control the results ([1],[4],[5],[14]).

Motivated by all these developments, in this paper we introduce a new operator and use it to define two new subclasses of analytic functions. In Section 3, we study some of their properties and then apply concept of third-order fuzzy differential subordination to obtain several results.

Let $\mathfrak{H}$ be the class of all normalized analytic functions in U and $\mathfrak{H}[a, k]$ be the subclass of $\mathfrak{H}$ consisting of functions

$$\mathfrak{H}[a, k] = \{f(z) \in U : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \dots\}$$

Let $\mathcal{A} \subset \mathfrak{H}$ be the class of functions analytic in U and has the Taylor-Maclaurin series of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1,1)$$

Where $U = \{z \in \mathbb{C} : |z| < 1\}$, $\bar{U} = \{z \in \mathbb{C} (1,1) : |z| \leq 1\}$

And $\partial U = \{z \in \mathbb{C} : |z| = 1\}$.

Let functions $f, g \in \mathcal{A}$, where $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. When Hadamard product, or convolution of f and g denoted by

$(f * g)(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$$

For $0 < q < p \leq 1$, the (p, q) -differential operator [14] is defined as $D_{p,q} : \mathcal{A} \rightarrow \mathcal{A}$

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$$D_{p,q}f(z) = \frac{[f(pz) - f(qz)]}{[(p - q)z]}, z \neq 0$$

The (p, q) – number for $k \in N$ is

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}.$$

Letting $\lambda \geq 0$ and $\beta \geq 0$, we have

$$\begin{aligned} E_{p,q,\beta}^{\lambda}f(z) &= (\Omega_{p,q,\beta}^{\lambda} * f)(z) \\ &= \left(1 - \frac{2}{1 + e^{-\beta}}\right)z + \frac{2}{1 + e^{-\beta}}[(1 - \lambda)f(z) + z\lambda D_{p,q}f(z)] \\ &= z + \sum_{k=2}^{\infty} \frac{2}{1 + e^{-\beta}}[1 + \lambda([k]_{p,q} - 1)]a_k z^k \end{aligned}$$

For $\delta \in N_0$, We define new operator $E_{p,q}^{\lambda,\delta}: \mathcal{A} \rightarrow \mathcal{A}$ as δ – times Hadamard product of $E_{p,q,\beta}^{\lambda}$ gives by

$$\begin{aligned} E_{p,q}^{\lambda,\delta}f(z) &= \underbrace{\Omega_{p,q,\beta}^{\lambda}(z) * \Omega_{p,q,\beta}^{\lambda} \dots * \Omega_{p,q,\beta}^{\lambda}}_{\delta\text{-times}} * f(z) \\ &= z + \sum_{k=2}^{\infty} \left(\frac{2}{1 + e^{-\beta}}[1 + \lambda([k]_{p,q} - 1)]\right)^{\delta} a_k z^k, \end{aligned} \quad (1.2)$$

Where $\Omega_{p,q,\beta}^{\lambda}(z) = \frac{2}{1 + e^{-\beta}}[1 + \lambda([k]_{p,q} - 1)]$ and $\beta, \lambda \geq 0, \delta \in N_0$.

Remark 1.1. The operator $E_{p,q}^{\lambda,\delta}$ defined by (1.2) reduces (p, q) – differential operator when $\delta = \lambda = 1, \beta = 0$ [14], to the q -differential operator when $p = 1, \beta = 0$ and $\lambda = \delta = 1$ [4], to the Al-Oboudi operator when $p = 1, q \rightarrow 1^-, \beta = 0$ [2], and to the operator when $\delta = 1, \lambda = 0$ [8].

Using this operator, we now define two new subclasses

Definition 1.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}^{\delta,\beta}(\mu, p, q, \lambda)$ of starlike functions with respect to operator $E_{p,q}^{\lambda,\delta}$ if

$$\operatorname{Re} \left\{ \frac{z \left(E_{p,q}^{\lambda,\delta} f(z) \right)'}{E_{p,q}^{\lambda,\delta} f(z)} \right\} > \mu, 0 \leq \mu < 1, z \in U. \quad (1.3)$$

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Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}^{\delta,\beta}(\mu, p, q, \lambda)$ of convex functions with respect to operator $E_{p,q}^{\lambda,\delta}$ if

$$\operatorname{Re} \left\{ 1 + \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)''}{\left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'} \right\} > \mu, \quad 0 \leq \mu < 1, \quad z \in U. \quad (1.4)$$

The classes $\mathcal{M}^{0,\beta}(\mu, p, q, \lambda)$ and $\mathcal{H}^{0,\beta}(\mu, p, q, \lambda)$ were introduced by Robertson [12].

2. Preliminaries

For the purpose of proving our new findings on the third-order fuzzy differential subordination, we shall present some essential definitions and lemma in this section.

Definition 2.1.[7] Let W denote the set of functions w that are analytic and univalent on the set $\bar{U} - N(w)$ where $\bar{U} = U \cup \partial U$ and $N(q) = \{\zeta \in \partial U: \lim_{z \rightarrow \zeta} w(z) = \infty\}$ and are such that

$$\min |w'(z)| = \tau > 0 \text{ for } \partial U - N(w)$$

Definition 2.2.[9] Let $\mathcal{X}$ be nonempty set. An application $F: \mathcal{X} \rightarrow [0,1]$ called fuzzy set.

Definition 2.3.[10] A pair (M, F_M) , where $F_M: \mathcal{X} \rightarrow [0,1]$ and $M = \{x \in \mathcal{X}: 0 < F_M(x) < 1\}$ is called fuzzy subset of $\mathcal{X}$ and F_M is called the membership function of the fuzzy set (M, F_M) . One can also denote $M = \operatorname{supp}(M, F_M)$.

Definition 2.4.[10] Let $B \subset \mathcal{C}$ and let $z_0 \in B$ be a fixed point and let the functions f, g be holomorphic in U . The function f is said to be fuzzy subordinate to function g and $f(z) <_F g(z)$ if there exists a function $F: \mathcal{C} \rightarrow [0,1]$ such that

- (1) $f(z_0) = g(z_0)$,
- (2) $F(f(z)) \leq F(g(z)), z \in B$.

Definition 2.5.[3] Let Φ be a function, $w \in W$ and $k \geq 2$. Functions $\psi: \mathcal{C}^4 \times U \rightarrow \mathcal{C}$ are called admissible functions and belong to the so-called class of admissible functions denoted by $\Psi_k[\Phi, w]$ if the admissibility condition

$$\psi(r, s, t, u; z) \notin \Phi,$$

Is satisfied when

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$$r = w(\zeta), s = k \zeta w'(\zeta), \operatorname{Re}\left\{\frac{t}{s} + 1\right\} \geq k \left[\operatorname{Re}\left\{1 + \frac{\zeta w''(\zeta)}{w'(\zeta)}\right\}\right],$$

$$\text{And } \operatorname{Re}\left\{\frac{u}{s}\right\} \geq k^2 \operatorname{Re}\left\{\frac{\zeta^2 w''(\zeta)}{w'(\zeta)}\right\}$$

Where $z \in U, \zeta \in \partial U - N(w)$.

If $\Phi \neq \mathbb{C}$ is also a simply connected domain, there exists a function h which maps conformal the unit disc satisfying the condition $h(0) = \psi(a, 0, 0, 0; 0)$ and h is a conformal mapping of U onto Φ we denote this class by $\psi \in \Psi\kappa[h, w]$.

Definition 2.6. [11] Let $h \in \mathfrak{H}(U)$ and $\psi: \mathbb{C}^4 \times U \rightarrow \mathbb{C}, \psi \in \Psi\kappa[h, w]$. A function $\varphi \in \mathfrak{H}(U)$ is called solution for a third-order fuzzy differential subordination if it satisfies.

$$F_{\psi(\mathbb{C}^4 \times U)} \psi(\varphi(z), z\varphi'(z), z^2\varphi''(z), z^3\varphi'''(z); z) \leq F_{h(U)} h(z)$$

$$\text{i.e. } \psi(\varphi(z), z\varphi'(z), z^2\varphi''(z), z^3\varphi'''(z); z) <_F h(z).$$

Lemma 2.1. [11] Taking a function $\varphi \in \mathfrak{H}[a, k]$ when $k \geq 2$, consider a function $w \in W(a)$ satisfying the conditions:

$$\frac{\zeta w''(\zeta)}{w'(\zeta)} \geq 0, \left| \frac{z\varphi'(z)}{w'(\zeta)} \right| < k$$

Where $z \in U, \zeta \in \partial U - N(w)$. Also consider $F: \mathbb{C} \rightarrow [0, 1]$. If $\psi \in \Psi\kappa[h, w]$ and $\psi(\varphi(z), z\varphi'(z), z^2\varphi''(z), z^3\varphi'''(z); z) \in \mathfrak{H}(U)$, then we have

$$\psi(\varphi(z), z\varphi'(z), z^2\varphi''(z), z^3\varphi'''(z); z) <_F h(z),$$

Implies

$$\varphi(z) <_F w(z).$$

3. Main Results

In this section, coefficient estimate for

$\mathcal{M}^{\delta, \beta}(\mu, p, q, \lambda)$ and $\mathcal{H}^{\delta, \beta}(\mu, p, q, \lambda)$ are first derived as follows:

Theorem 3.1. Let $f \in \mathcal{A}$, if the following inequality holds:

$$\sum_{k=2}^{\infty} \left[\frac{k-\mu}{1-\mu} \right] \left(\frac{2}{1+e^{-s}} [1 + \lambda([k]_{p,q} - 1)] \right)^{\delta} |a_k| \leq 1, \quad (3.1)$$

For $0 \leq \mu < 1$, then f belongs to $\mathcal{M}^{\delta, \beta}(\mu, p, q, \lambda, \delta)$.

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Proof: To prove, $f \in \mathcal{M}^{\delta,\beta}(\mu, p, q, \lambda, \delta)$, it suffices to show that

$$\left| \frac{\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} - 1}{\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} + 1 - 2\mu} \right| < 1$$

From (1.2) we have:

$$E_{p,q,\beta}^{\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} \mathcal{B}_k a_k z^k, (E_{p,q,\beta}^{\lambda,\delta} f)'(z) = 1 + \sum_{k=2}^{\infty} k \mathcal{B}_k a_k z^{k-1},$$

Where $\mathcal{B}_k = \left(\frac{2}{1+e^{-s}} [1 + \lambda([k]_{p,q} - 1)] \right)^{\delta}$. Then

$$\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} - 1 = \frac{\sum_{k=2}^{\infty} (k-1) \mathcal{B}_k a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} \mathcal{B}_k a_k z^{k-1}}$$

Taking moduli and invoking the triangle inequality, we obtain

$$\left| \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} (k-1) \mathcal{B}_k |a_k|}{1 + \sum_{k=2}^{\infty} \mathcal{B}_k |a_k|}$$

Using condition (3.1), we get:

$$\sum_{k=2}^{\infty} \mathcal{B}_k |a_k| \leq \frac{1-\mu}{k-\mu} \leq 1-\mu$$

And after simplification we obtain:

$$\operatorname{Re} \left\{ \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right\} > \mu.$$

Consequently, f belongs to $\mathcal{M}^{\delta,\beta}(\mu, p, q, \lambda, \delta)$ as required. $\square$

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Theorem 3.2. Let $f \in \mathcal{A}$, if the following inequality holds:

$$\sum_{k=2}^{\infty} \left[\frac{k(k-\mu)}{1-\mu} \right] \left(\frac{2}{1+e^{-s}} [1 + \lambda([k]_{p,q} - 1)] \right)^{\delta} |a_k| \leq 1. \quad (3.2)$$

For $0 \leq \mu < 1$, then f belongs to $\mathcal{H}^{\delta,\beta}(\mu, p, q, \lambda, \delta)$.

Proof: The proof follows similar lines as Theorem 3.1 by

considering $1 + \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))''}{(E_{p,q,\beta}^{\lambda,\delta} f(z))'}$ and noting that differentiation brings an extra factor k . $\square$

Finally, we present several results related to the third-order fuzzy differential subordination with respect to operator $E_{p,q,\beta}^{\lambda,\delta}$.

Theorem 3.3. Let $f \in \mathcal{A}$ and $\lambda, \beta \geq 0, \delta \in N_0, 0 < q < p \leq 1$. Let $w(z)$ be convex univalent in U with $w(0) = 1$. If

$$\begin{aligned} & \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} + \nu_1 z \left(\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right)' + \nu_2 z^2 \left(\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right)'' + \\ & \nu_3 z^3 \left(\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right)''' \prec_F h(z), \end{aligned} \quad (3.3)$$

where $h(z)$ is convex univalent in U with $h(0) = 1$ and $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$. Then

$$\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \prec_F w(z), \quad (3.4)$$

Function w satisfying (3.4) is the fuzzy best dominant.

Proof: To prove this result, we introduce the auxiliary function

$$\varphi(z) = \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)}, \quad (3.5)$$

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because $f \in A$ and $E_{p,q,\beta}^{\lambda,\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{1+e^{-\beta}} [1 + \lambda([k]_{p,q} - 1)] \right)^{\delta} a_k z^k$, it follows that $\varphi(z)$ is analytic in U and $\varphi(0) = 1$.

Now we compute the derivatives of $\varphi(z)$. By logarithmic differentiation of (3.5) we obtain

$$\frac{z\varphi'(z)}{\varphi(z)} = 1 + \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)''}{\left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'} - \varphi(z).$$

From this relation, after some simplification we can express $z\varphi'(z)$, $z^2\varphi''(z)$ and $z^3\varphi'''(z)$ in terms of $E_{p,q,\beta}^{\lambda,\delta} f(z)$ and its derivatives. Substituting these expressions into $\mathcal{G}(z)$ given in (3.3), we see that $\mathcal{G}(z)$ reduces to the form:

$$\mathcal{G}(z) = \varphi(z) + v_1 z\varphi'(z) + v_2 z^2\varphi''(z) + v_3 z^3\varphi'''(z), \quad (3.6)$$

Let $\psi(r, s, t, u; z) = r + v_1 s + v_2 t + v_3 u$, where $r = \varphi(z)$, $s = z\varphi'(z)$, $t = z^2\varphi''(z)$ and $u = z^3\varphi'''(z)$. Then the third-order fuzzy differential subordination $\mathcal{G}(z) \prec_F h(z)$ becomes $\psi(\varphi(z), z\varphi'(z), z^2\varphi''(z), z^3\varphi'''(z); z) \prec_F h(z)$. (3.7)

Since $\varphi(z)$ is analytic with $\varphi(0) = 1$, $v_1, v_2, v_3 \in \mathbb{C}$, $w(z)$ is convex with $w(0) = 1$ and $h(z)$ is convex with $h(0) = 1$, the admissibility condition $\psi \in \Psi\kappa[h, w]$ is satisfied by Definition 2.5. Hence, by Lemma 2.1, we have $\varphi(z) \prec_F w(z)$ in U . Finally, using the definition of $\varphi(z)$ from (3.5), we obtain (3.4) and q is the fuzzy best dominant. Thus we obtain the required conclusion. $\square$

By assigning specific values to the parameter v_1, v_2, v_3 we obtain the following original result.

Corollary 3.1. Let $f \in \mathcal{A}$ and $\lambda, \beta \geq 0$, $\delta \in \mathbb{N}_0$, $0 < q < p \leq 1$. Let $q(z)$ be convex univalent in U with $q(0) = 1$. If

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$$\frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} + \nu z \left(\frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right)' \prec_F h(z),$$

Where $h(z)$ is convex univalent in U with $h(0) = 1$ and $\nu \in \mathbb{C}$. Then

$$\frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \prec_F q(z).$$

Proof: The result follows from Theorem 3.3 by taking $\nu_1 = \nu$ and $\nu_2 = \nu_3 = 0$. $\square$

As an application of Theorem 3.3, we deduce a sufficient condition for the operator $E_{p,q,\beta}^{\lambda,\delta}$ to belong to the class of starlike functions of order μ .

Corollary 3.2. Let $f \in \mathcal{A}$ and $0 \leq \mu < 1$, $\lambda, \beta \geq 0$, $\delta \in N_0$, $0 < q < p \leq 1$. Let $M = 1 + |\nu_1| + 2|\nu_2| + 6|\nu_3|$. If

$$\left| \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} + \nu_1 z \left(\frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right)' + \nu_2 z^2 \left(\frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right)'' + \nu_3 z^3 \left(\frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right)''' - 1 \right| < \frac{1-\mu}{M},$$

Where $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$. For all $z \in U$, then

$$\operatorname{Re} \left\{ \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right\} > \mu. \text{ Hence } E_{p,q,\beta}^{\lambda,\delta} f \text{ is starlike in } U.$$

Proof: Set $h(z) = 1 + \frac{1-\mu}{M}z$ and $w(z) = 1 + \frac{1+(1-2\mu)z}{1-z}$ in Theorem 3.3. Since $w(z)$ is convex univalent and $h(U) \subset w(U)$, all

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conditions Theorem 3.3 are satisfied. Thus $\frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \prec_F 1 +$

$\frac{1+(1-2\mu)z}{1-z}$ which is equivalent to $Re \left\{ \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))'}{E_{p,q,\beta}^{\lambda,\delta} f(z)} \right\} > \mu$. Thus

$E_{p,q,\beta}^{\lambda,\delta} f$ is starlike in U . $\square$

Theorem 3.3 provides a general framework for starlikeness. By applying a similar technique to the function $\varphi(z) = 1 + \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))''}{(E_{p,q,\beta}^{\lambda,\delta} f(z))'}$, we now establish an analogous result for convexity.

Theorem 3.4. Let $f \in \mathcal{A}$ and $\lambda, \beta \geq 0, \delta \in N_0, 0 < q < p \leq 1$. Let $w(z)$ be convex univalent in U with $w(0) = 1$. If

$$1 + \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))''}{(E_{p,q,\beta}^{\lambda,\delta} f(z))'} + v_1 z \left(1 + \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))''}{(E_{p,q,\beta}^{\lambda,\delta} f(z))'} \right)' + v_2 z^2 \left(1 + \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))''}{(E_{p,q,\beta}^{\lambda,\delta} f(z))'} \right)'' + v_3 z^3 \left(1 + \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))''}{(E_{p,q,\beta}^{\lambda,\delta} f(z))'} \right)''' \prec_F h(z)$$

Where $h(z)$ is convex univalent in U with $h(0) = 1$ and $v_1, v_2, v_3 \in \mathbb{C}$. Then

$$1 + \frac{z(E_{p,q,\beta}^{\lambda,\delta} f(z))''}{(E_{p,q,\beta}^{\lambda,\delta} f(z))'} \prec_F w(z).$$

And w is the fuzzy best dominant.

Corollary 3.3. Let $f \in \mathcal{A}$ and $0 \leq \mu < 1, \lambda, \beta \geq 0, \delta \in N_0, 0 < q < p \leq 1$. Let $M = 1 + |v_1| + 2|v_2| + 6|v_3|$. If.

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$$\left| 1 + \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)''}{\left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'} + v_1 z \left(1 + \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)''}{\left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'} \right)' \right. \\ \left. + v_2 z^2 \left(1 + \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)''}{\left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'} \right)'' + v_3 z^3 \left(1 + \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)''}{\left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'} \right)''' - 1 \right| < \frac{1-\mu}{M} z$$

Where $v_1, v_2, v_3 \geq 0$. Then

$$\operatorname{Re} \left\{ 1 + \frac{z \left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)''}{\left(E_{p,q,\beta}^{\lambda,\delta} f(z) \right)'} \right\} > \mu.$$

Hence $E_{p,q,\beta}^{\lambda,\delta} f$ is convex in U .

4. Conclusions

This paper presents a novel operator $E_{p,q,\beta}^{\lambda,\delta}$ defined on analytic functions. By using this operator, we defined new subclasses of analytic functions. The main results include coefficient estimates and third-order fuzzy subordination theorems. Future research may focus on generalizing the operator to broader classes of functions, examining its Hankel determinant and studying third-order fuzzy super ordination. Moreover, the application of fuzzy subordination concepts shows promise in areas that deal with imprecise data, such as control systems, modeling, and decision-making processes.

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